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ON THE MOMENT MATRIX $E(n)$

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1 Introduction and preliminaries

Consider a collection of complex numbers

$$\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \gamma_{1,2n-1}, \dots, \gamma_{2n-1,1}, \gamma_{2n,0},$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$. The *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane \mathbb{C} such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu(z) \quad (0 \leq i + j \leq 2n); \quad (1.1)$$

μ is called a *representing measure* for γ . This truncated complex moment problem has been well-established ([CuF1], [CuF2], [JLLL]).

We recall first some notation from [CuF1] and [CuF2]. For $n \geq 1$, let $m \equiv m(n) = (n+1)(n+2)/2$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \dots, Z^n, \bar{Z}Z^{n-1}, \dots, \bar{Z}^{n-1}Z, \bar{Z}^n.$$

We define $M(n) := M(n)(\gamma) \in \mathcal{M}_m(\mathbb{C})$ as follows: for $0 \leq k + l \leq n, 0 \leq i + j \leq n$, the entry in row $\bar{Z}^k Z^l$ and column $\bar{Z}^i Z^j$ is $M(n)_{(k,l)(i,j)} = \gamma_{l+i,j+k}$. These matrices come from Bram-Halmos characterization for a cyclic operator T satisfying $\gamma_{ij} = (T^{*i} T^j x_0, x_0)$, where x_0 is a cyclic vector for T (cf. [Br] or [Con]). So it is nature to consider moment matrices corresponded by Embry characterization for subnormality of such operators (cf. [Em] or [Con]). We will write such matrices by $E(n)$.

Consider a collection of complex numbers

$$\gamma \equiv \{\gamma_{ij}\} (0 \leq i+j \leq 2n, |i-j| \leq n) \text{ with } \gamma_{00} > 0 \text{ and } \gamma_{ji} = \bar{\gamma}_{ij}.$$

For $n \in \mathbb{N}$, let

$$m = m[n] = \left(\left[\frac{n}{2} \right] + 1 \right) \left(\left[\frac{n+1}{2} \right] + 1 \right).$$

For $A \in M_m(\mathbb{C})$, we first introduce the following order on the rows and columns of A : $1, Z, Z^2, \bar{Z}Z, Z^3, \bar{Z}Z^2, Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2, Z^5, \dots$. We denote the entry of A in row $\bar{Z}^k Z^l$ and column $\bar{Z}^i Z^j$ by $A_{(k,l)(i,j)}$. If $n = 2k$, $k = 1, 2, \dots$, let

$$\mathcal{SP}_n = \{p(z, \bar{z}) = a_{00} + a_{01}z + a_{02}z^2 + a_{11}\bar{z}z + a_{03}z^3 + a_{12}\bar{z}z^2 + \dots + a_{kk}\bar{z}^k z^k\};$$

if $n = 2k + 1$, $k = 0, 1, 2, \dots$, let

$$\mathcal{SP}_n = \{p(z, \bar{z}) = a_{00} + a_{01}z + a_{02}z^2 + a_{11}\bar{z}z + a_{03}z^3 + a_{12}\bar{z}z^2 + \dots + a_{k,k+1}\bar{z}^k z^{k+1}\},$$

where $a_{ij} \in \mathbb{C}$. It is clear that \mathcal{SP}_n is a subspace of \mathcal{P}_n , the vector space of all complex polynomials in z, \bar{z} of total degree $\leq n$. For $p \in \mathcal{SP}_n$, let $\hat{p} = [a_{00}, a_{01}, \dots, a_{kk}]^T$ (which means the transposed) or $[a_{00}, a_{01}, \dots, a_{k,k+1}]^T$ in \mathbb{C}^m . We define a sesquilinear form $\langle \cdot, \cdot \rangle_A$ on \mathcal{SP}_n by $\langle p, q \rangle_A := \langle A\hat{p}, \hat{q} \rangle$ ($p, q \in \mathcal{SP}_n$). In particular, $\langle \bar{z}^i z^j, \bar{z}^k z^l \rangle_A = A_{(k,l)(i,j)}$, for $0 \leq i+j \leq n, i \leq j$ and $0 \leq k+l \leq n, k \leq l$. For γ , we define the moment matrix $E(n) \equiv E(n)(\gamma) \in M_m(\mathbb{C})$ as follows: $E(n)_{(k,l)(i,j)} := \gamma_{l+i, j+k}$.

The following provides a motivation to study the truncated moment theory of $E(n)$, whose proof can be found in [JKLP].

Theorem 1.1. *Let S be a contractive subnormal operator with a cyclic vector x_0 in \mathcal{H} and let $\gamma_{ij} = (S^{*i}S^j x_0, x_0)$. Then the following assertions are equivalent:*

- (i) $M(n) \geq 0$ for any $n \in \mathbb{N}$;
- (ii) $E(n) \geq 0$, for any $n \in \mathbb{N}$;
- (iii) *there exists a positive Borel measure μ supported in the complex plane \mathbb{C} such that*

$$\gamma_{ij} = \int_{\mathbb{D}} \bar{z}^i z^j d\mu(z) \quad \text{for any } i, j \in \mathbb{N} \cup \{0\},$$

where \mathbb{D} is the closed unit disc in \mathbb{C} .

We may give the following conjecture, as in [CuF1].

Conjecture 1.2. *Let $\gamma \equiv \{\gamma_{ij}\} (0 \leq i+j \leq 2n, |i-j| \leq n)$ be a truncated moment sequence. The following statements are equivalent:*

- (i) γ has a rank $E(n)$ -atomic representing measure;
- (ii) $E(n) \geq 0$ and $E(n)$ admits a flat extension $E(n+1)$.

In this article, we will consider the conjecture concretely and give the double flat extension theorem.

2 Moment matrices $E(n)$ and representing measures

If μ is the representing measure for γ , then $\langle E(n)\hat{p}, \hat{p} \rangle = \int |p(z, \bar{z})|^2 d\mu$, for $p(z, \bar{z}) \in \mathcal{SP}_n$. Hence $E(n) \geq 0$. But the converse implication is not always true (see Example 2.2 below). We first introduce an analogous statement with that of $M(n)$. For $p \in \mathcal{SP}_n$, let $\mathcal{Z}(p) = \{z \in \mathbb{C} : p(z, \bar{z}) = 0\}$.

Lemma 2.1. ([JKLP]) *Let $\gamma \equiv \{\gamma_{ij}\} (0 \leq i+j \leq 2n, |i-j| \leq n)$. Assume that γ has a representing measure μ . For $p \in \mathcal{SP}_n$, $\text{supp } \mu \subseteq \mathcal{Z}(p) \iff p(Z, \bar{Z}) = 0$.*

Example 2.2. Consider

$$M := E(3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1+i & 0 \\ 0 & 1 & 0 & 0 & 1+i & 2 \\ 0 & 0 & 2 & 1-i & 0 & 0 \\ 1 & 0 & 1+i & 2 & 1+i & 0 \\ 1-i & 1-i & 0 & 1-i & 4 & 2(1-i) \\ 0 & 2 & 0 & 0 & 2(1+i) & 4 \end{bmatrix}.$$

It is easy to show that $E(3)$ is positive and $\text{rank } E(3) = 3$. In fact, $\det([M]_4) = \det([M]_5) = \det M = 0$, where $[M]_k$ is the left upper $k \times k$ submatrix. Furthermore, we have

$$\begin{aligned} \bar{Z}Z &= 1 + \frac{1-i}{2}Z^2, \\ \bar{Z}Z^2 &= 2Z, \\ Z^3 &= (1+i)1 + (1+i)Z. \end{aligned}$$

$$\begin{aligned}
p_1(z, \bar{z}) &= 1 + \frac{1-i}{2}z^2 - \bar{z}z, \\
p_2(z, \bar{z}) &= 2z - \bar{z}z^2, \\
p_3(z, \bar{z}) &= (1+i) + (1+i)z - z^3.
\end{aligned}$$

Then $\mathcal{Z}(p_1, p_2, p_3) = \{z \in \mathbb{C} : p_i(z, \bar{z}) = 0, i = 1, 2, 3\} = \emptyset$. Thus for the given moment sequence γ in $E(3)$, there is no representing measure for γ .

Theorem 2.3. ([CuF1]) *If $\gamma \equiv \{\gamma_{ij}\}(0 \leq i+j \leq 2n)$ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.*

The above theorem produces Conjecture 1.2.

We showed this conjecture is true in the case of even numbers [JKLP]. We can provide a counter example for Conjecture 1.2 in the case $n = 3$.

Example 2.4. (Example 2.2 revisited) Since $\text{rank } E(2) = \text{rank } E(3) = 3$, $E(3)$ is flat. If $E(3)$ admits a flat extension $E(4)$, then

$$Z^4 = (1+i)Z + (1+i)Z^2, \quad \bar{Z}Z^3 = 2Z^2. \quad (2.1)$$

From the first equality of (2.1), we obtain $\gamma_{34} = 2$, and from the second equality of (2.1), we obtain $\gamma_{34} = 0$. Hence $E(3)$ has no flat extension of $E(4)$.

So we have the following theorems in sharpness whose proof can be found in [JKLP].

Theorem 2.5. *Let $n \geq 2$. If γ is double flat (i.e., $\text{rank } E(n) = \text{rank } E(n-2)$) and $E(n) \geq 0$, then $E(n)$ admits a unique flat extension of the form $E(n+1)$.*

Theorem 2.6. *The truncated complex moment sequence $\gamma \equiv \{\gamma_{ij}\}(0 \leq i+j \leq 2n, |i-j| \leq n)$ has a rank $E(n)$ -atomic representing measure if and only if $E(n) \geq 0$ and $E(n)$ admits a double flat extension $E(n+2)$, i.e., $\text{rank } E(n) = \text{rank } E(n+2)$.*

Finally, we give the following example that affirm Theorem 2.6.

Example 2.7. Let

$$E(2) = \begin{bmatrix} 1 & 0 & i & 1 \\ 0 & 1 & 1+i & 1-i \\ -i & 1-i & 3 & -3i \\ 1 & 1+i & 3i & 3 \end{bmatrix}.$$

Then $E(2)$ admits a double flat extension $E(4)$ as the following

$$E(4) = \begin{bmatrix} E(3) & B^* \\ B & C \end{bmatrix},$$

where

$$E(3) = \begin{bmatrix} 1 & 0 & i & 1 & i-1 & 1+i \\ 0 & 1 & 1+i & 1-i & 3i & 3 \\ -i & 1-i & 3 & -3i & 4+4i & 4-4i \\ 1 & 1+i & 3i & 3 & -4+4i & 4+4i \\ -1-i & -3i & 4-4i & -4-4i & 11 & -11i \\ 1-i & 3 & 4+4i & 4-4i & 11i & 11 \end{bmatrix},$$

$$B = \begin{bmatrix} -3 & -4-4i & -11i & -11 & 15(1-i) & -15(1+i) \\ -3i & 4-4i & 11 & -11i & 15(1+i) & 15(1-i) \\ 3 & 4+4i & 11i & 11 & -15(1-i) & 15(1+i) \end{bmatrix},$$

$$C = \begin{bmatrix} 41 & -41i & -41 \\ 41i & 41 & -41i \\ -41 & 41i & 41 \end{bmatrix}.$$

In fact, $\text{rank } E(2) = \text{rank } E(4) = 2$. Since

$$\begin{cases} z^2 = i + (1+i)z, \\ \bar{z}z = 1 + (1-i)z, \end{cases}$$

we obtain two atoms $z_0 = (1 - \sqrt{3})(1+i)/2$ and $z_1 = (1 + \sqrt{3})(1+i)/2$. According to

$$\begin{bmatrix} 1 & 1 \\ z_0 & z_1 \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} = \begin{bmatrix} \gamma_{00} \\ \gamma_{01} \end{bmatrix},$$

we have $\rho_0 = \frac{1+\sqrt{3}}{2\sqrt{3}}$, $\rho_1 = \frac{-1+\sqrt{3}}{2\sqrt{3}}$. Thus we obtain the representing measure $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1}$.

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